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REVIEW

Theory and phenomenology for a variety of classical and quantum phase transitions

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Abstract In this review, we first introduce recent progress in the mathematical structure of the three-dimensional Ising model, from the points of view of topologic, algebraic and geometric aspects. Then we discuss in turn Anderson localization due to disorder and then first- and second-order metal-insulator transitions, depending on electron correlation, with and without a magnetic field. Finally, we make intimate contact with the phase diagram showing the equilibrium between low temperature regimes of the magnetically induced Wigner electron solid and the so-called Laughlin electron liquid in the two-dimensional case.

Keywords Ising model · Anderson localization · Metal-insulator transitions · Classical phase transitions · Quantum phase transitions · Laughlin electron liquid · Wigner electron solid

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1 Introduction

Understanding the critical phenomena in various materials has proven to be one of the most difficult challenges in physics. Though the Ising model is a very simple model (though difficult still to solve for dimensionality $D = 3$, see below), it can reveal the essential nature of the critical behavior at/near the critical point. In spite of the pioneering work of Onsager [1] on the two-dimensional (2D) version, some aspects of the three-dimensional (3D) case remain however controversial. Much progress has occurred on the theory of phase transitions, following early ideas of Kadanoff [2] and Widom [3], which were brought to full fruition in the renormalization group studies of Wilson [4]. Notwithstanding this progress, there remain crucial outstanding problems like, for instance, the exact solution of the 3D Ising model. On the one hand, the root of the difficulties for the exact solution of the 3D Ising model is believed to be topologic, which should be dealt with globally. On the other hand, all the well-accepted theories or approaches, including the low- and high-temperature expansions [5,6], the renormalization group [4,7] and the Monte Carlo method [8], take into account only the local environments of a spin. The critical exponents of the 3D Ising model were proposed by Zhang [9]; see also [10], in a paper which was quite clearly stated to be based on two conjectures, associated with a rotation in a 4D space and weight factors on the eigenvectors, respectively. After publication of Ref. [9], objections raised in [11–16] have focused mainly on the disagreement between the conjectured solution and the series expansions, but have not commented on the topology-based approach underlying the derivation. However, as pointed out in [17–21], all the well-known theorems [22–33] for the convergence of the high-temperature series are proved only for $\beta(= 1/k_BT) > 0$, not for infinite temperature ($\beta = 0$). The infinite temperature limit has been never touched on in these theorems [22–33] cited in [11–16], since there exists a singularity at $\beta = 0$. Recently, the algebraic part of the quaternion approach used in Ref. [9] was reformulated in terms of the quaternionic sequence of Jordan algebras to examine further the geometrical aspects of simple orthorhombic Ising lattices [34], and fractals and chaos related to these 3D Ising lattices were also investigated [35–37]. The controversy existing in this field is mainly due to a fact that, as we face up to nature we like the blind feel an elephant by touching. Therefore, it is very important to study further the mathematical structure of the 3D Ising model, in order to understand more deeply the problems.

In this review, in Sect. 2.1, we therefore introduce briefly recent progress in the mathematical structure of the 3D Ising model [21], which certifies the validity of Zhang's two conjectures [9]. In Sect. 2.2, we then discuss briefly singularities at/near infinite temperature, which are of importance for evaluating whether the well-known high-temperature expansions can serve as a standard for judging the validity of the conjectured exact solution of the 3D Ising model. In Sects. 3 and 4, we discuss in turn Anderson localization, a second-order transition induced by disorder, and then first- and second-order metal-insulator transitions, with and without a magnetic field, driven by electron correlation in a two-dimensional electron assembly. Finally, we make intimate contact with the phase diagram showing the equilibrium between low temperature regimes of the magnetically induced Wigner electron solid and the so-called Laughlin electron liquid in the two-dimensional case.

2 Mathematical structure of the 3D Ising model

2.1 Topologic, algebraic and geometric aspects

Recently, one of us (ZDZ) gave an overview of the mathematical structure of the 3D Ising model [21], from the viewpoints of topologic, algebraic and geometric aspects. Here we summarize briefly the results obtained in [21].

It is well-known that the main difficulties for solving explicitly the 3D Ising model are caused by so-called internal factors in the transfer matrix **V** [13,18], which are topologic. Two kinds of contribution to the partition function can be distinguished: namely (1) those reflecting the local arrangement of spins and (2) the remainder reflecting the non-local behaviour of the knots. These two types of contribution are subsumed into the transfer matrix **V** for the 3D Ising model [9,17,18]. Therefore, the procedure for the exact solution of the 3D Ising model involves the smoothing of knots/crosses in the transfer matrix **V**. Topologically, there are two choices for smoothing a given crossing ($\times$), and thus there are 2^N states of a diagram with N crossings [9,38–40]. The bracket state summation, $\langle K\rangle = \sum_\sigma \langle K|\sigma\rangle d^{\|\sigma\|}$, is an analog of a partition function in discrete statistical mechanics [9,38–40]. Here σ runs over all the states of K. $\mathrm{d} = -\mathrm{A}^2 - \mathrm{B}^2$, with A, B and d being commuting algebraic variables. According to the topological theory, one could transform from the basis of $\langle\chi\rangle$ and $\langle\chi^{-1}\rangle$ (see Eq. (2.1) below) to the basis of $\langle \overset{\cup}{\underset{\cap}{}} \rangle$ and $\langle\rangle\langle\rangle$ by a transformation, and vice versa. The transformation can be written as [21,39,40]:

$$\begin{bmatrix} \langle\chi\rangle \\ \langle\chi^{-1}\rangle \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \left\langle \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right\rangle \\ \langle\rangle\langle\rangle \end{bmatrix} \tag{2.1}$$

The bracket with $\mathrm{B} = \mathrm{A}^{-1}, \mathrm{d} = -\mathrm{A}^2 - \mathrm{A}^{-2}$ is invariant under the Reidemeister moves II and III [39,40].[1] Therefore, a (complex) matrix representing the unitary transformation (a rotation) may always exist, no matter how complicated the knots or links are. The 3D interacting Ising system with non-trivial knots or links requires naturally the existence of the additional dimension (say, 'time'). After smoothing, there will be no crossing in the new matrix $V' \equiv V'_4 \cdot V'_3 \cdot V'_2 \cdot V'_1$ [9], which precisely includes the topologic contribution to the partition function and becomes diagonalizable.

Meanwhile, a diagram of a knot or link can usually be interpreted as an abstract tensor diagram, by using an oriented diagram and associating two matrices R_{cd}^{ab} and $\overline{R}_{cd}^{ab}$ with two types of crossing [39,40]. Then, any oriented link diagram K can be mapped to a specific contracted abstract tensor T(K). If the matrices R and $\overline{R}$ satisfy channel unitary, cross-channel unitary and the Yang–Baxter equation [41–43], then T(K) is a regular isotopy invariant for oriented diagram K. It is worthwhile noting that the Yang–Baxter equation corresponds to a Reidemeister Move of type III. The

[1] A Reidemeister move refers to one of three local moves on a link diagram. Each move operates on a small region of the diagram and is one of three types: I. Twist and untwist in either direction. II. Move one loop completely over another. III. Move a string completely over or under a crossing.

Yang–Baxter equation in the form of the universal R-matrix can be represented as [39]:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.2}$$

with $R_{12} = \sum_s e_s \otimes e^s \otimes 1$, $R_{13} = \sum_s e_s \otimes 1 \otimes e^s$ and $R_{23} = \sum_s 1 \otimes e_s \otimes e^s$. The Yang–Baxter equation and its generalization ensure the commutativity of the transfer matrices and the integrability of exactly solvable models of statistical mechanics [40–43]. However, the Yang–Baxter equation can be utilized only to solve the 2D models, and its 3D analogue is the so-called tetrahedron equation, or generalized Yang–Baxter equation [44–52]. Decomposition and rearrangement of a tetrahedron (of a rhombic dodecahedron) show how to deal with the topological problem of the 3D models [46] by disconnection/fusion of the crossings. Satisfying the tetrahedron relation guarantees the commutativity of the transfer matrices and the integrability of the 3D Ising models. The disconnection/fusion of the crossings causes the emerging of the phase factors in the 3D Ising model. The Jordan algebra and the Jordan-von Neumann–Wigner procedures guarantee the commutative relations [34–37,53], and thus the existence of the tetrahedron relation.

The procedure for explicitly solving the 3D Ising model is related with Lie algebras/Lie group, via quaternions, Pauli matrices, special unitary group SU(2), rotation matrices SO_2 (**R**), and special orthogonal group SO(3) [21]. We have to deal with the 3D Ising model in much larger Hilbert space by introducing the additional dimension, because the operators generate much larger Lie algebras, due to the appearance of nonlocal behaviour (knots) [9]. In the 3D Ising model, one obtains a paravector by adding the fourth dimension to form quaternion eigenvectors, giving a result which corresponds to the Clifford or geometric algebra $C\ell_3$. The quaternion approach developed in Ref. [9] can be made more elegant and simple by the use of Clifford structures and the P. Jordan structures [34–37]. The natural appearance of the multiplication $A \circ B = \frac{1}{2}(AB + BA)$ in Jordan algebras instead of the usual matrix multiplication AB satisfies the desire for commutative subalgebras of the algebra constructed in Ref. [9] and for their combinatorial properties. It is known that the unit quaternions can be thought of as a choice of a group structure on the 3-sphere S^3 that gives the group Spin(3), which is isomorphic to SU(2) and also to the universal cover of SO(3). Therefore, the complexified quaternion basis constructed in Ref. [9] for the 3D Ising model represents naturally a rotation in the 4D space (a (3 + 1)-dimensional spacetime). Performing the fourfold integration of the partition function of the 3D Ising model meets the requirement of taking the time average [9,54]. This is related closely with well-developed theories, for example, complexified quaternion [55], quaternionic quantum mechanics [56–58], and quaternion and special relativity [59]. In a recent work [60], we proposed that quaternion-based functions developed in Ref. [9] for the 3D Ising model can be utilized to study the conformal invariance in dimensions three. The 3D conformal transformations can be decomposed into three 2D conformal transformations, where the Virasoro algebra still works, but only for each complex plane of quaternionic coordinates in the complexified quaternionic Hilbert space.

As a generalization of the geometric relations obtained in Ref. [1] for the 2D Ising model, the geometric relations obtained in Ref. [9], i.e., eqns. (29)–(32) of Ref. [9], for

the 3D Ising model, are those in a hyperbolic 3-sphere, which can be represented in the 4D Poincaré disk (ball) model. The 3-sphere has a natural Lie group structure given by quaternion multiplication, which is consistent with the quaternion eigenvectors constructed in Ref. [9] for the 3D Ising model. According to observations in Ref. [9], the duality transformations of the simple orthorhombic Ising models are between the edges and their corresponding faces of the two dual orthorhombic lattices. It is expected that the duality between other 3D lattices should be related also with the edges and their corresponding faces of the dual lattices [21]. The duality relation between two dual tetrahedron lattices, or alternatively, between a tetrahedron lattice and a 3D honeycomb lattice could map a low-temperature (high-temperature) model on the tetrahedron lattice to a high-temperature (low-temperature) one on the 3D honeycomb lattice. Moreover, the balance between the exchange energy and thermal activity of the Ising model is related with the geometric duality, fractal and quasicrystals. Some geometrical connections should exist between the golden (or silver) ratio (the critical point of the most symmetric 3D (or 2D) lattice), Fibonacci (or Octonacci) number, 3D tenfold (or 2D eightfold) quasicrystal, and fractal of flower (or branch) type [21]. The weight factors for the 3D Ising model were interpreted as geometric (or topologic) phases [17], similar to those in Berry phase effect, Aharonov-Bohm effect, Josephson effect,, the Quantum Hall effects, etc. The novel phases appeared in the 3D Ising model [9,17] can be understood further, in connection with quantum field theory and gauge models [21].

2.2 Singularities of the free energy of the 3D Ising system at/near infinite temperature

In this subsection, we briefly discuss singularities of the free energy of the system at/near infinite temperature, since it is useful to judge whether the high-temperature expansions can serve as a standard for the exact solution of the 3D Ising model.

We have pointed out [19,21] that the singularities of the reduced free energy βf, the free energy per site f and the free energy F of the 3D Ising model differ at $\beta = 0$. The definition of the free energy per site f loses physical significance at $T = \infty$, and one has to face directly the total free energy F to study the singularities of the system. One needs to discuss not only the zeros, but also the poles of the partition function Z for complete information of the system, specially, for the singularities at infinite temperature. This is because the intrinsic characters of the singularities of the zeros (and the poles) at infinite temperature are quite different from those at finite temperatures.

As pointed out already in Refs. [17–21], all the well-known theorems [22–33] for the convergence of the high-temperature series of the 3D Ising model are rigorously proved only for $\beta(= 1/k_BT) > 0$, not for infinite temperature ($\beta = 0$). These theorems either deal with the reduced free energy βf (or βp) or set $\beta = 1$ for their proof. For equivalence between βf and f (or that between βp and p), setting $\beta = 1$ equalizes to $T = 1/k_B \neq \infty$; it corresponds to $\beta \neq 0$. This indicates clearly that such a rigorous proof is valid only for high, but still finite, temperature. Lebowitz and Penrose indicated clearly in the abstract of their paper [30] that their proof for the analytics of the free

energy per site and the distribution function of the Ising model is for $\beta > 0$. Though Lebowitz and Penrose claimed that the hard-core systems are analytic in β at $\beta = 0$, actually, their proof concerns βp (the series (4) of Ref. [30]), not p itself. Thus the hard-core systems are analytic in β only for high but finite temperature. The hard-core potential is defined by $\varphi(r) = +\infty$ for $r \leq a$, and $\varphi(r) < \infty$ for $r > a$, where a is a positive constant ($a > 0$) [30]. The Ising ferromagnet is isomorphic to a lattice gas with an attractive interaction potential with $\varphi(0) = +\infty$, and $\varphi(r) \leq 0$ for $r \neq 0$ [30]. Even when we treat the Ising model as a special case (with $a = 0$) of the hard-core models, the claim of Lebowitz and Penrose is applied to high but finite temperature, since their proof concerns βp.

Perk claimed in his recent article [15] that his Theorem 2.9 proves rigorously the analyticity of the reduced free energy βf in terms of β at $\beta = 0$. Actually, some mathematical tricks had been performed in his procedure to avoid the difficulty of singularities at $\beta = 0$, which first appear in Definition 1.4, defining the free energy per site f_N and its infinite system limit f by eqn (6), but in form of $-\beta f_N$ [15]. Then Lemma 2.5 went on perpetrating the difficulty, discussing the singularity of βf_N, and finally to prove Theorem 2.9 'rigorously' for βf [15].

In the 3D Ising model there indeed exist three singularities [18–21]: (1) $H = 0$, $\beta = \beta_c$; (2) $H = \pm i\beta, \beta \to 0$; (3) $H = 0, \beta = 0$. The third singularity has physical significance [18–21]: The 3D Ising system experiences a change from a 'non-interaction' state at $\beta = 0$ to an interacting state at $\beta > 0$. This change of state is similar to a switch turning off/on all the interactions at/near infinite temperature, resulting in a change of the topologic structures and the corresponding phase factors [9,18–21]. The singularities of the free energy F and the free energy per site f at $\beta = 0$ support that two different forms for infinite temperature and finite temperatures could exist for the high-temperature series expansions of the free energy per site f, as revealed in Ref. [9].

3 Critical exponents for Anderson localization due to disorder

We turn next to a briefer discussion of the critical exponents for Anderson localization due to disorder. The early work of Mott and Twose [61] demonstrated conclusively that, in one dimension, all the electronic states are localized, even for an arbitrarily small degree of disorder. But the crucial work was that of Anderson [62] in three dimensions, who drew attention specifically to the absence of diffusion in certain random lattices. Subsequently, a number of group (see, for example, Dancz, Edwards and March [63]; this work is summarized by March and Angilella [64]) argued that in two dimensions (as well as in one: see [61]), all the electron states were localized for arbitrarily small degree disorder. However, Anderson [62] showed that above a certain degree of disorder in three dimensions, the electronic states were localized.

With this short Introduction to older studies, we turn to more recent work. Here, a pioneering contribution was that, García-García [65] studied analytically the metal-insulator transition in a disordered conductor by combining the self-consistent theory of localization with the one parameter scaling theory.

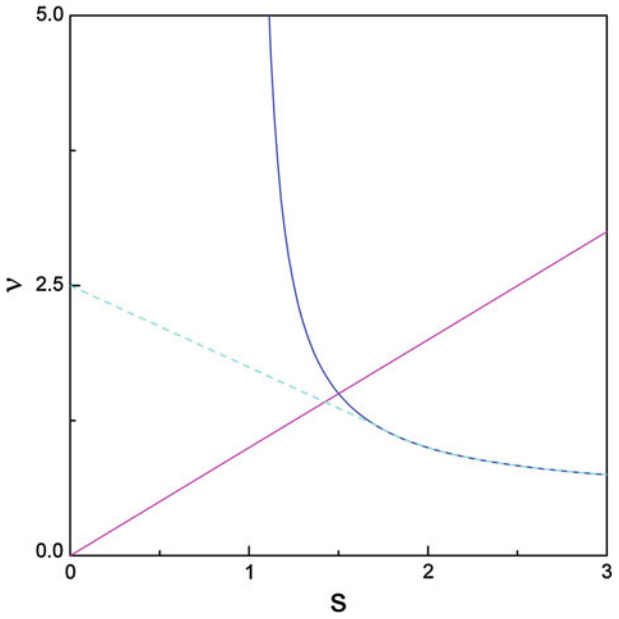

Fig. 1 *Continuous curve* shows plot of the localization length exponent ν versus the conductivity critical exponent s, from $(2s-2)\,\nu = s$, which is a semiclassical approximation in which equation of $s = \frac{d}{2}$ approximates s [67]. Lower, *dashed sketch* shows refinement of the semiclassical result proposed in which the upper continuous curve is the asymptote for sufficiently large s. The major difference between the two curves lies in the behaviour of s/ν at $d = 2$. In the semiclassical case, s/ν is zero at $s = 1 (d = 2)$ because ν becomes infinite there [see $s = \frac{2\nu}{2\nu-1}$]. As in the *dashed curve*, $s/\nu = d - 2$ is satisfied by s becoming zero at $d = 2$, and ν remaining finite (5/2). If the value 5/2 assumed proves not universal, the inequality $\nu > 2/d$ shows that $\nu(s = 0) > 1$. Note that the straight line in Fig. 1 is $\nu = s$

Using Wegner's result relating critical exponents s and ν for conductivity and localization length, respectively via dimensionality d and that for ν given by García–García [65], we derived in [66,67] what we term a semiclassical relation for ν in terms of s which is independent of dimensionality. Forming the ratio s/ν versus d from the above relations, $s/\nu = 0$ at $d = 2$ is due to a singularity in the semiclassical relation for ν. We argued that, in reality, $s/\nu = 0$ results from s being zero at $d = 2$. Finally we conjecture that (1) Wegner's prediction that $s/\nu = 1$ when $d = 3$ and (2) ν tends to 1/2 at large s, are both insensitive to interactions.

In Fig. 1, the continuous curve shows plot of the localization length exponent ν versus the conductivity critical exponent s, from eqn (5) of Ref. [67], which is a semiclassical approximation in which eqn (4) of Ref. [67] approximates s. The lower, dashed sketch in Fig. 1 shows refinement of the semiclassical result proposed in which the upper continuous curve is the asymptote for sufficiently large s. The major difference between the two curves in Fig. 1 lies in the behaviour of s/ν at $d = 2$. In the semiclassical case, s/ν is zero at $s = 1 (d = 2)$ because ν becomes infinite there (see eqn (7) of Ref. [67]). In the behaviour illustrated as the dashed curve sketched in Fig. 1, $s/\nu = d - 2$ from eqn (1)

of Ref. [67] is satisfied by s becoming zero at d = 2, and ν remaining finite (5/2).

4 Metal-insulator transitions, including Wigner electron crystallization, with and without magnetic fields

As the third and final topic of this review, we shall focus on metal-insulator transitions driven by electron correlation. This takes us back at least to the ideas of Wigner [68] in 1934, who was concerned at that time with the quantum-mechanical jellium model of a metal. To take the alkali metals Na and K, it was already thought that, at atmospheric pressure and low temperatures, these could be usefully modeled by smearing out the Na^+ ions, say, into a uniform neutralizing unresponsive background in which the (3s) conduction electrons moved under the influence of their mutual Coulombic repulsion.

4.1 Thermodynamics of Wigner electron crystallization

Following Parrinello and March [69], let us begin by examining the case of constant pressure p and temperature T.

Denoting the Gibbs free energy by G, the equilibrium condition for the electron liquid to crystal transition is

$$G_1 = G_2 \tag{4.1}$$

For changes ΔE, ΔS and ΔV in internal energy, entropy and volume, respectively across the transition, Eq. (4.1) yields

$$\Delta E = T\Delta S - p\Delta V. \tag{4.2}$$

For the Coulomb system under discussion, the virial theorem takes the form

$$2K + U = 3pV, \tag{4.3}$$

K denoting kinetic and U potential energy. Thus

$$p\Delta V = \frac{2}{3}\Delta K + \frac{1}{3}\Delta U \tag{4.4}$$

and hence it follows from Eqs. (4.4) and (4.2) that

$$T\Delta S = \frac{5}{3}\Delta K + \frac{4}{3}\Delta U. \tag{4.5}$$

(a) Melting curve in classical limit

The classical limit of Eq. (4.5) is readily found to be

$$\Delta S = \frac{4}{3}\frac{\Delta U}{T}, \tag{4.6}$$

while Eq. (4.4) yields

$$\Delta V = \frac{1}{3}\frac{\Delta U}{p}. \tag{4.7}$$

Employing the Clausius–Clapeyron equation for a first-order phase transition, namely,

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V}, \tag{4.8}$$

leads to the melting curve in this classical regime as

$$p = AT^4. \tag{4.9}$$

(b) Zero temperature predictions

Returning to Eq. (4.5) and inserting T = 0 we find

$$\Delta E = \Delta K + \Delta U = \frac{1}{5}\Delta U \tag{4.10}$$

and

$$\Delta U = -\frac{1}{5p}\Delta V \tag{4.11}$$

Turning next to the case of V and T constant, we have the equilibrium condition as $H_1 = H_2$, with H the Helmholtz free energy. From the virial result (4.3) we have further that

$$V\Delta p = \frac{2}{3}\Delta E - \frac{1}{3}\Delta U \tag{4.12}$$

and from the equilibrium condition it follows that

$$\Delta E = T\Delta S \tag{4.13}$$

These Eqs. (4.12) and (4.13) then yield the following limiting results:

(a) Classical

$$\Delta p = \frac{1}{3V}\Delta U \tag{4.14}$$

and

$$\Delta S = \frac{\Delta U}{T} \tag{4.15}$$

The melting curve following from Eqs. (4.14) and (4.15), was already known, for example in Pollock and Hansen' s work [70] as

$$V = AT^{-3} \tag{4.16}$$

or

$$T = \chi \rho^{\frac{1}{3}} \tag{4.17}$$

where ρ denotes the number density, proportional to V^{-1}, while

$$\chi = (Ze)^2 \left(\frac{4}{3}\pi\right)^{1/3} \Big/ 158 k_B, \tag{4.18}$$

Ze being the charge on the point ions.

(b) Case of zero temperature

The T = 0 limit becomes, in case of constant V

$$\Delta E = 0 \tag{4.19}$$

and

$$\Delta p = -\frac{1}{3V}\Delta U \tag{4.20}$$

d-dimensional results

Following again Parrinello and March [69], the above thermodynamics can be extended to the case of a Coulomb assembly in d dimensions. Then, following for instance Toulouse's work [71], which does not however consider non-zero pressure, the virial theorem becomes

$$2T + (d-2)U = dpV. \tag{4.21}$$

The most interesting case after d = 3 is when d = 4. Then Eq. (4.21) reduces to

$$E = 2pV. \tag{4.22}$$

Hence the melting curve in the density-temperature (ρ, T) plane follows as

$$T = C\rho^{1/2} \tag{4.23}$$

As Parrinello and March stress [69], Eq. (4.23) is, in fact, fully quantal. From Eq. (4.23), the average interelectronic spacing r_s for crystallization, r_s^* say, is infinite. This follows also from scaling considerations. For a change of scale $r \to r' = r/s$, the jellium (homogeneous electron gas) Hamiltonian H becomes H':

$$H = -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + \frac{e^2}{2}\sum_{i\neq j}\frac{1}{|r_i - r_j|^{d-2}} \to H' \tag{4.24}$$

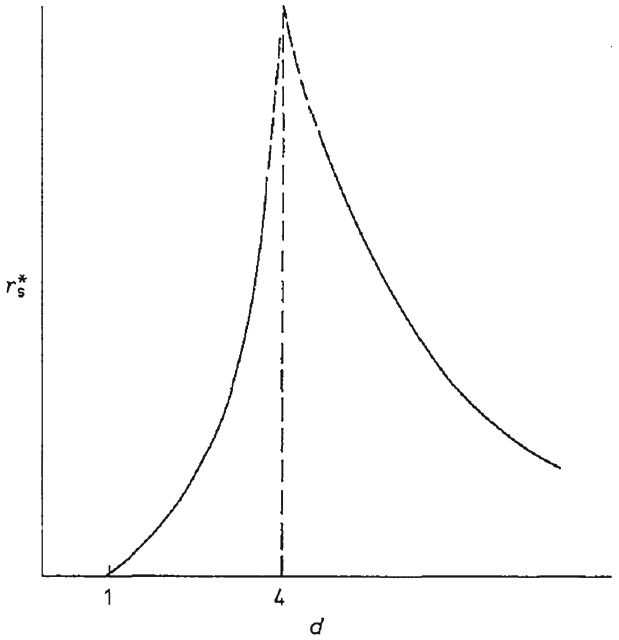

Fig. 2 Critical r_s for Wigner crystallization as a function of dimensionality d (schematic) [69]

The transformed Hamiltonian in Eq. (4.24) reads then

$$H' = s^2 \left(-\frac{\hbar^2}{2m} \sum_i \nabla_i^2 + \frac{e^2}{2} s^{-\epsilon} \sum_{i \neq j} \frac{1}{|r_i - r_j|^{d-2}} \right) \tag{4.25}$$

where $\epsilon = 4 - d$. Thus, for the high density limit s $\rightarrow \infty$, we approach the free particle case for $\epsilon > 0$ and the Wigner case for $\epsilon < 0$.

We therefore have the qualitative plot for r_s^* as a function of d shown in Fig. 2. The point d = 1 corresponds to $r_s^* = 0$. This is due to the absence of phase transitions in one dimension and for strong repulsive (Coulomb) interactions in this case an ordered state must result (compare [72]).

4.2 Equilibrium between a quantum liquid and a two-dimensional Wigner electron solid in a perpendicular magnetic field

The localization of electrons in impure semiconductors was treated by Durkan et al. [73] with emphasis on transport properties of highly compensated InSb in an applied magnetic field. In this study, the proposal was made that Wigner electron crystallization could be aided by means of localization brought about by strong applied magnetic fields. This proposal of a magnetically induced Wigner solid (MIWS) has subsequently been confirmed by experiments of Andrei et al. [74] in a two-dimensional

GaAs/AlGaAs heterojunction. Their main findings have been verified and extended by the photoluminescence experiments of Buhmann et al. [75].

Motivated by these studies, and especially by the desire to understand the melting curve of the MIWS, Lea et al. [76] have given the thermodynamics of such melting in a magnetic field. Following, for example, Pippard's account [77], for the two-dimensional Wigner electron crystal under discussion the melting temperature T_m of the first-order transition to an electron liquid must obey the thermodynamic relation

$$\left(\frac{\partial T_m}{\partial H}\right)_\Omega = -\frac{\Delta m}{\Delta S}, \tag{4.26}$$

where Ω now defines the volume. If we denote the crystal phase by the subscript C and the liquid phase by the subscript L, then $\Delta m = m_L - m_C$ is the change of magnetization in melting, while $\Delta S = S_L - S_C$ is the corresponding change in the entropy. It is conventional to characterize H by the Landau level filling factor ν, given in terms of the areal electron density n and the magnetic field applied perpendicular to the electron layer by

$$\nu = nhc/eH. \tag{4.27}$$

At this point, we summarize the schematic phase diagram, proposed by Buhmann et al. [75], and following Lea et al. [76] in Fig. 3. Four crystal phases, denoted $C_1 - C_4$, are shown interspersed with the liquid phase at the filling factors corresponding to the fractional quantum Hall effect at $\nu = \nu_q = 1/q$ with $q = 5, 7$ and 9. The final solid phase drawn by Buhmann et al. [75] ends at a critical filling factor of $\nu_c = 0.28 \pm 0.02$. These experimental results prompted Lea, March and Sung [76] to generalize the thermodynamics of Wigner crystallization, already considered by Parrinello and March [69] as reviewed earlier. This led to the form, from Eqs. (4.26) and (4.27) above, that

$$\left(\frac{\partial T_m}{\partial \nu}\right)_\Omega = \left(\frac{H}{\nu}\right)\frac{\Delta m}{\Delta S} \tag{4.28}$$

The properties following from Eq. (4.28) are then summarized in the caption to Fig. 3.

The essence of Fig. 3 is then explained in terms of the change in magnetization Δm across the melting curve by Lea, March and Sung [76] and the action shown in the schematic Fig. 3 being centred on the diamagnetism of the so-called Laughlin liquid phase in equilibrium with the Wigner electron crystal. Lea et al. [76] explained the schematic form in terms of an anyon model, appropriate, of course, in two dimensions. These authors noted that this field dependence of Δm was reminiscent of the de Haas-von Alphen effect at integral ν values.

This prompts us to cite the subsequent work of Wu et al. [78]. These authors gave a detailed theoretical treatment of the thermal activation of quasiparticles and the thermodynamic observables in the Laughlin liquid focused on by Lea et al. [76]. The important conclusion of Wu et al. [78] is that what are usually thought of as different, but equivalent pictures (anyon, composite Fermion and composite Boson) of the effect

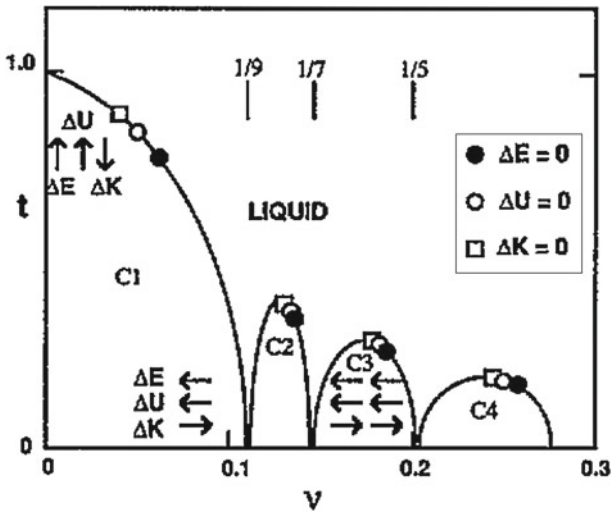

Fig. 3 Schematic phase diagram showing the four crystal phases, C1–C4, and the reentrant liquid phase at $\nu = 1/9, 1/7$ and 1/5. The *symbols* mark the points ΔE, ΔU and $\Delta K = 0$ as deduced from thermodynamics [76]. The *arrows* show the direction of ΔE, ΔU and ΔK across the phase boundary nearest to the arrows. The arrows in phases C2 and C4 match those shown in phase C3

under discussion can exhibit significant differences at finite temperature, though they all prove to be equivalent at $T = 0$. Then the final sentence of Wu et al. [78] reads, and we quote 'In particular, it is more desirable that these theoretical predictions would be put to experimental tests, if the tremendous difficulties in measuring thermodynamic quantities of a thin layer of electron gas could be overcome someday'.

It is therefore the more remarkable that the existing experiments of Andrei et al. [74] and Buhmann et al. [75], as interpreted by Lea et al. [76], allow detailed thermodynamic properties to be discussed from the equilibrium between a Laughlin liquid and a Wigner electron solid (see also [79,80]).

4.3 Phenomenological theory of first- and second- order metal-insulator (MI) transitions in three dimensions at $T = 0$

Below the phenomenological theory proposed by March, Suzuki and Parrinello [81] (below MSP) will be outlined. The point to be stressed first is that the order parameter in their phenomenology is the discontinuity, say q, in the single-particle occupation probability at the Fermi surface.

First, let us consider the case of the second-order MI transition in a half-filled Hubbard band. q, as shown in Ref. [81], turns out then to have a critical exponent of unity. MSP also expose the relation to Gutzwiller's variational treatment. Additionally, the enhancement of the spin susceptibility by the Hubbard interaction will also be discussed below.

To avoid repetition, let us therefore include in the energy expansion in low orders of q also the magnetization m and the applied field h. Then the energy expansion

proposed by MSP reads:

$$E(m,q) = E_0 + am^2 - hm + E_1 q + E_2 q^2 + \cdots + eqm^2 \tag{4.29}$$

Minimizing with respect to m, one readily finds that

$$2am + 2eqm = h \tag{4.30}$$

and hence the spin susceptibility χ is

$$\chi = m/h = \frac{1}{2\,[a(U) + eq]} \tag{4.31}$$

Here, we have used a notation which makes explicit the fact that the quantity a will depend of the magnitude of the Hubbard U. This, in turn, is the energy cost of placing two electrons with opposite spin on the same lattice site.

If the transition is second order at a critical value of U, denoted below as U_c, then $q \to 0$ as $U \to U_c$. Then the physically important question as to the enhancement of the magnetic susceptibility χ as the MI transitions is approached rests on the behavior of a(U) as a function of the Hubbard interaction. Provided $a \to 0$ as $U \to U_c$ as $(1 - U/U_c)$ or faster, then MSP demonstrate that

$$\chi \propto \frac{U_c}{U_c - U} \tag{4.32}$$

It is of practical interest to note here the related discussion by Brinkman and Rice [82] in relation to experiments on the solid V_2O_3, which indeed show enhancement of χ by U in the metallic phase in the proximity of the MI transition.

The same phenomenology used above for a second-order MI transition can also be applied to the melting of a Wigner electron crystal, the melting phase transition being first order. The gist of the results already discussed in some detail above again follows from the MSP phenomenology.

Subsequently, we note that Chapman and March [83] have generalized the MSP phenomenology to non-zero temperature to gain some insight into the behavior of the spin susceptibility χ as a function of T along the liquid-vapor coexistence curve towards the critical point for liquid metals (e.g. Cs). As a byproduct of this analysis, Chapman and March [83] were able to estimate for liquid Cs the way the jellium prediction of the discontinuity q in the momentum distribution at the Fermi surface depended on the electron-ion interaction: of course not dealt with in the jellium model. Their conclusion, which has some support from NMR experiments, is that q can be increased by a factor of 2 or more by the presence of electron-ion interaction, in liquid metallic Cs.

5 Brief concluding remarks and possible future direction

Emphasis has been given in this review to the statistical thermodynamics of a number of Hamiltonians.

The first of these is that of the 3D Ising model. Because the detail rapidly proliferates, we have thought it important in this review to consider recent progress in the mathematical structure. Thus topological, algebraic and geometric aspects are focused on as concisely as possible in Sect. 2.1. By analyzing the mathematical structure, we have illustrated the following facts for the 3D Ising model [21]: the complexified quaternion basis constructed for the 3D Ising model represents naturally the rotation in a $(3+1)$-dimensional space-time, which is consistent with the fourfold integrand of the partition function by taking the time average. A unitary transformation with a matrix being a spin representation in $2^{n\cdot l\cdot o}$-space corresponds to a rotation in $2^{n\cdot l\cdot o}$-space, which serves to smooth all the crossings in the transfer matrices, contributes as the non-trivial topologic part of the partition function and changes the wave functions by complex phases ϕ_x, ϕ_y, and ϕ_z of the 3D Ising model. A tetrahedron relation would ensure the commutativity of the transfer matrices and the integrability of the 3D Ising model, and its existence is guaranteed also by the Jordan algebra and the Jordan-von Neumann–Wigner procedures. The analysis of this Ising model is already argued by studying conformal invariance in three dimensions [19]. The 3D conformal transformations can be decomposed into three 2D such transformations, where it is important that Virasoro algebra is still appropriate. The arguments presented bear on, and support, the two conjectures made by ZDZ in [9], which lead to the complete set of critical exponents for the 3D Ising model. The rich features revealed in [21] for the mathematical structure of the 3D Ising model just show a corner of mathematical iceberg below the sea level of the physical phenomena [1–37,54,60,84–96].

Much more briefly we have next considered Anderson localization as driven by disorder. The important semiclassical discussion by García-García [65] is first emphasized and then in Fig. 1 the changes we have proposed to the critical exponents [66,67] are displayed. Further work to confirm, or if necessary to improve Fig. 1 is still called for.

Then in Sect. 4 a treatment of metal-insulator transition is given, based as starting point on the so-called jellium Hamiltonian (see Eq. (4.24) in D dimensions). This is followed by a largely thermodynamic discussion of the equilibrium between a quantum (Laughlin) liquid and a two-dimensional Wigner electron solid in a perpendicular magnetic field. Close contact is thereby made with experiments carried out in [74] and [75] using a two-dimensional GaAs/AlGaAs heterojunction. Finally, a phenomenological theory of both first- and second-order metal-insulator transitions is summarized, but now by invoking the Hubbard Hamiltonian [see Eqs. (4.31) and (4.32) for the magnetic susceptibility χ as a function of the Hubbard interaction] in three dimensions at $T = 0$. In each of the areas considered in this overview, we stress the continuing importance of the closest possible interaction between theory and experiment.

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